BOREL STRUCTURE OF THE SPECTRUM OF A CLOSED OPERATOR

PIOTR NIEMIEC

Abstract. For a linear operator T in a Banach space let $\sigma_p(T)$ denote the point spectrum of T, $\sigma_{p[n]}(T)$ for finite $n > 0$ be the set of all $\lambda \in \sigma_p(T)$ such that $\dim \ker(T - \lambda) = n$ and let $\sigma_{p[\infty]}(T)$ be the set of all $\lambda \in \sigma_p(T)$ for which $\ker(T-\lambda)$ is infinite-dimensional. It is shown that $\sigma_p(T)$ is $\mathcal{F}_\sigma$, $\sigma_{p[\infty]}(T)$ is $\mathcal{F}_{\sigma\delta}$ and for each finite n the set $\sigma_{p[n]}(T)$ is the intersection of an $\mathcal{F}_\sigma$ and a $\mathcal{G}_\delta$ set provided T is closable and the domain of T is separable and weakly σ-compact. For closed densely defined operators in a separable Hilbert space $\mathcal{H}$ more detailed decomposition of the spectra is done and the algebra of all bounded linear operators on $\mathcal{H}$ is decomposed into Borel parts. In particular, it is shown that the set of all closed range operators on $\mathcal{H}$ is Borel.

2010 MSC: Primary 47A10, Secondary 54H05, 28A05.
Key words: spectrum, point spectrum, Hilbert space, reflexive Banach space, Borel set, closed operator, weak topology.

1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space. As it is easily seen, any subset of the complex plane is the point spectrum of an unbounded linear operator acting on $\mathcal{H}$. To see this, note that the dimension of $\mathcal{H}$ as a vector space is equal to the power of the continuum; take a Hamel basis $\{e_s\}_{s\in\mathbb{R}}$ of $\mathcal{H}$ and any function $\psi\colon \mathbb{R} \to \mathbb{C}$, define $A\colon \mathcal{H} \to \mathcal{H}$ by the rule $Ae_s = \psi(s)e_s$, and observe that the point spectrum of A coincides with the image of the function f (see also [7, Example 2]). So, non-Borel subsets of $\mathbb{C}$ may be the point spectra of certain linear operators acting on $\mathcal{H}$. It is also worth while to mention that every bounded subset of $\mathbb{C}$ is the point spectrum of a bounded normal (even diagonal) operator on a nonseparable Hilbert space. However, all densely defined linear operators in separable Hilbert spaces whose point spectra are non-Borel which appear in all existing examples in the literature are nonclosed. It was therefore quite natural to ask for such an example in the class of closed operators. In the recent paper we will show that such an example does not exist. More generally, we will prove the following result (see Theorem 3.5 and Theorem 4.2):

> *Let X be a (real or complex) Banach space and T be a closable operator in X whose domain is separable and weakly σ-compact. Then the point spectrum of T is $\mathcal{F}_\sigma$*

and for each $n \geqslant 1$ the set of all scalars λ such that the kernel of $T - \lambda$ is n-dimensional is $\mathcal{F}_{\sigma\delta}$.

In particular, the above theorem applies to the point spectrum of a closed operator in $\mathcal{H}$, which solves the problem posed in [7] (see Question after Example 2 there). Partial solution of this issue for certain classes of closed operators in Hilbert spaces may be found e.g. in [13, Corollary 7]. In Example 3.9 we give a simple example of a bounded densely defined operator in $\mathcal{H}$ whose point spectrum coincides with an arbitrarily given uncountable subset of the unit disc. The problem whether the point spectrum of a bounded operator in a separable nonreflexive Banach space is $\mathcal{F}_\sigma$ we leave as open.

Since the celebrated paper by Cowen and Douglas [3] was published, the point spectra of bounded linear operators are widely investigated and those operators which have these spectra rich have attracted a growing interest, see e.g. [9], [17], [6] or [7] and references there.

Notation. In this paper Banach spaces are real or complex and they need not be separable, and $\mathbb{K}$ is the field of real or complex numbers. By an *operator in* a Banach space X we mean any linear function from a linear subspace of X into X and such an operator is *on* X if its domain is the whole space. For an operator T in X, $\mathcal{D}(T)$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}}(T)$ denote the domain, the kernel, the range and the closure of the range of T, respectively. For a scalar $\lambda \in \mathbb{K}$, $T - \lambda$ stands for the operator given by the rules: $\mathcal{D}(T-\lambda) = \mathcal{D}(T)$ and $(T-\lambda)(x) = Tx - \lambda x$ $(x \in \mathcal{D}(T))$. The *point spectrum* of T, $\sigma_p(T)$, is the set of all eigenvalues of T; that is, $\sigma_p(T)$ consists of all scalars $\lambda \in \mathbb{K}$ such that $\mathcal{N}(T - \lambda)$ is nonzero. The operator T is *closed* iff its graph $\Gamma(T) := \{(x, Tx)\colon\ x \in \mathcal{D}(T)\}$ is a closed subset of $X \times X$. T is *closable* iff the norm closure of $\Gamma(T)$ in $X \times X$ is the graph of some operator in X. Whenever E and F are Banach spaces, $\mathcal{B}(E, F)$ stands for the space of all bounded operators from the whole space E to F. A subset of a topological space is said to be *σ-compact* if it is the union of a countable family of compact subsets of the space. A *weakly σ-compact* subset of a Banach space is a set which is σ-compact with respect to the weak topology of the space. A *Borel* subset of a topological space is a member of the σ-algebra generated by all open sets in the space. Adapting the notation proposed by A.H. Stone [14], a subset of a topological space which is the intersection of an open and a closed set is called by us (*of type*) $\mathcal{F} \cap \mathcal{G}$. Similarly, any set being the intersection of a $\mathcal{G}_\delta$ and an $\mathcal{F}_\sigma$ set will be said to be $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$.

2. Arbitrary closed operator in a Banach space

In this and next sections X denotes a Banach space.

Recall that a *Souslin* set is a metrizable space which is the continuous image of a separable complete metric space (see the beginning

of Chapter XIII of [8] and Theorem XIII.1.1 there, or Appendix in [15]). Souslin subsets of separable complete metric spaces are *absolutely measurable*, i.e. whenever A is a Souslin subset of a separable complete metric space Y and μ is a nonnegative σ-finite Borel measure on Y, there are Borel subsets B and C of Y such that $B \subset A \subset C$ and $\mu(C \setminus B) = 0$ (see e.g. [15, Theorem A.13]).

We begin with

2.1. **Proposition.** *Let T be a closable operator in X whose domain is the image of a separable Banach space under a bounded operator. The point spectrum of T is a Souslin subset of $\mathbb{K}$. In particular, $\sigma_p(T)$ is absolutely measurable.*

Proof. Let E be a separable Banach space and $S \in \mathcal{B}(E, X)$ be such that $S(E) = \mathcal{D}(T)$. Passing to the quotient of E by $\mathcal{N}(S)$, we may assume that S is one-to-one. Since T is closable, the operator $C = TS\colon E \to X$ is bounded (by the Closed Graph Theorem). Notice that $\sigma_p(T) = \{\lambda \in \mathbb{K}\colon\ \mathcal{N}((T-\lambda)S) = \mathcal{N}(C-\lambda S) \neq \{0\}\}$. Let W be the set of all $x \in E$ such that $\|x\| = 1$ and $Cx = \lambda(x)Sx$ for some $\lambda(x) \in \mathbb{K}$. We have obtained a function $\lambda\colon W \to \mathbb{K}$ with $\lambda(W) = \sigma_p(T)$. So, to finish the proof it suffices to show that W is closed and λ is continuous.

Suppose $x_n \in W$ and $x_n \to x \in X$ as $n \to \infty$. Then $\|x\| = 1$ and hence $Sx \neq 0$. We infer from this that the sequence $(\lambda(x_n))_{n=1}^{\infty}$ is bounded because $|\lambda(x_n)| = \frac{\|Cx_n\|}{\|Sx_n\|} \to \frac{\|Cx\|}{\|Sx\|}$ $(n \to \infty)$. Now if $(\lambda(x_{n_k}))_{k=1}^{\infty}$ is any subsequence which converges to some $w \in \mathbb{K}$, then $Cx = wSx$. This shows that $x \in W$ and $w = \lambda(x)$. So, W is closed and λ is continuous. $\square$

Now since the domain of a closed operator is the image of its graph under the projection onto the first factor, Proposition 2.1 applies to closed operators in separable Banach spaces and gives

2.2. **Corollary.** *The point spectrum of a closed operator in a separable Banach space is Lebesgue measurable.*

In the next section (Corollary 3.7) we shall prove that the point spectrum of a closed operator in a separable reflexive Banach space is Borel (even $\mathcal{F}_\sigma$). We do not know whether the assumption of reflexivity may be relaxed in this.

Question 1. Is the point spectrum of a bounded operator on a separable Banach space Borel?

3. Operators with weakly σ-compact domains

For an operator T in X, a subset K of X and a nonnegative real constant M, let

$$\Lambda_T(K, M) = \{w \in \mathbb{K}\colon\ \mathcal{N}(T - w) \cap K \neq \varnothing \text{ and } |w| \leqslant M\}.$$

Notice that $\Lambda_T(K,M)$ consists of all $w \in \mathbb{K}$ with $|w| \leqslant M$ provided $0 \in K$. The main tool of this section is the following

3.1. **Lemma.** *If T is a closable operator in X and K is a weakly compact subset of $\mathcal{D}(T)$, then the set $\Lambda_T(K,M)$ is compact for every $M \geqslant 0$.*

Proof. We may and do assume that $0 \notin K$. Let W be the set of all $x \in K$ for which there is (unique) $\lambda(x) \in \mathbb{K}$ such that $Tx = \lambda(x)x$ and $|\lambda(x)| \leqslant M$. Thus we have obtained a function $\lambda\colon W \to \mathbb{K}$. Since $\lambda(W) = \Lambda_T(K,M)$, it suffices to show that W is weakly compact and λ is continuous when W is equipped with the weak topology.

Let $\mathcal{X} = (x_\sigma)_{\sigma\in\Sigma}$ be a net in W which is weakly convergent to some $x \in K$. We need to show that $x \in W$ and $\lim_{\sigma\in\Sigma} \lambda(x_\sigma) = \lambda(x)$. If $(x_\tau)_{\tau\in\Sigma'}$ is a subnet of $\mathcal{X}$ such that $\lim_{\tau\in\Sigma'} \lambda(x_\tau) = w \in \mathbb{K}$, then $(x_\tau, Tx_\tau)_{\tau\in\Sigma'}$ is weakly convergent to (x, wx). Since the norm closure of $\Gamma(T)$ is weakly closed (and is the graph of some operator), we infer from this that $Tx = wx$ and thus $x \in W$ and $\lambda(x) = w$. Since $\lambda(W)$ is bounded and any convergent subnet of $(\lambda(x_\sigma))_{\sigma\in\Sigma}$ has the same limit, the latter net converges to $\lambda(x)$ and we are done. □

With use of the foregoing result we now easily prove the following

3.2. **Proposition.** *If T is a closable operator in X such that $\mathcal{D}(T)\setminus\{0\}$ is weakly σ-compact, then $\sigma_p(T)$ is $\mathcal{F}_\sigma$.*

Proof. Write $\mathcal{D}(T) \setminus \{0\} = \bigcup_{n=1}^\infty K_n$ with each K_n weakly compact, note that $\lambda \in \sigma_p(T)$ iff $\lambda \in \Lambda_T(K_n, m)$ for some natural n and m and apply Lemma 3.1. □

For applications of the above result, we need the next well known fact. For reader convenience, we give its short proof.

3.3. **Lemma.** *Every weakly compact subset of X which is separable in the norm topology is weakly metrizable.*

Proof. Let K be a separable weakly compact subset of X. Then $(K-K) \setminus \{0\}$ is separable as well and thus there is a sequence of linear functionals $f_n\colon X \to \mathbb{K}$ of norm 1 such that for every nonzero $z \in K - K$ there is n with $f_n(z) \neq 0$. Observe that then the family $\{f_n\}_{n\in\mathbb{N}}$ separates the points of K. Finally, since K is weakly compact, the formula $K \ni x \mapsto (f_n(x))_{n\in\mathbb{N}} \in \Delta^{\mathbb{N}}$ with $\Delta = \{w \in \mathbb{K}\colon\ |w| \leqslant 1\}$ defines a topological embedding of K (equipped with the weak topology) into the compact metrizable space $\Delta^{\mathbb{N}}$. □

Now we have

3.4. **Proposition.** *If $\mathcal{D}$ is a linear subspace of X, then $\mathcal{D}\setminus\{0\}$ is weakly σ-compact iff $\mathcal{D}$ is separable and weakly σ-compact as well.*

Proof. First assume that $\mathcal{D}$ is separable and $\mathcal{D} = \bigcup_{n=1}^{\infty} K_n$ with each K_n weakly compact. By Lemma 3.3, K_n is weakly metrizable and thus $L_n := K_n \setminus \{0\}$ is an $\mathcal{F}_\sigma$-subset of K_n with respect to the weak topology. So, $\mathcal{D} \setminus \{0\} = \bigcup_{n=1}^{\infty} L_n$ and each L_n is weakly σ-compact.

Conversely, if $\mathcal{D} \setminus \{0\}$ is weakly σ-compact, so is $\mathcal{D}$ and $\{0\}$ is weakly $\mathcal{G}_\delta$ in $\mathcal{D}$. Thus, by the definition of the weak topology, there are sequences $(f_n)_{n=1}^{\infty}$ and $(\varepsilon_n)_{n=1}^{\infty}$ of continuous linear functionals on X and of positive real numbers (respectively) such that $\{0\} = \mathcal{D} \cap \{x \in X\colon\ |f_n(x)| < \varepsilon_n,\ n = 1, 2, \ldots\}$. In particular, $\bigcap_{n=1}^{\infty} \mathcal{N}(f_n) \cap \mathcal{D} = \{0\}$ and therefore the function $\psi\colon \mathcal{D} \ni x \mapsto (f_n(x))_{n=1}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ is one-to-one. What is more, ψ is continuous with respect to the weak topology on $\mathcal{D}$ and the product one on $\mathbb{K}^{\mathbb{N}}$. So, ψ restricted to any weakly compact subset K of $\mathcal{D}$ is a topological embedding and therefore every such K is weakly separable. We conclude from this that $\mathcal{D}$ itself is weakly separable, being weakly σ-compact. Finally, if A is a countable weakly dense subset of $\mathcal{D}$ and $\mathcal{E}$ is the norm closure of the linear span of A, then $\mathcal{E}$ is separable and weakly closed. The latter implies that $\mathcal{D} \subset \mathcal{E}$ and we are done. □

It follows from Proposition 3.4 that Proposition 3.2 is equivalent to

3.5. **Theorem.** *If T is a closable operator in X whose domain is separable and weakly σ-compact, then $\sigma_p(T)$ is $\mathcal{F}_\sigma$.*

By Baire's theorem, a Banach space is weakly σ-compact iff it is reflexive. Thus Theorem 3.5 applies mainly to reflexive spaces. For example:

3.6. **Corollary.** *If T is a closable operator in X whose domain is separable and is the image of a reflexive Banach space under a bounded operator, then $\sigma_p(T)$ is $\mathcal{F}_\sigma$.*

Now argument used in the note just before Corollary 2.2 shows that

3.7. **Corollary.** *The point spectrum of a closed operator in a separable reflexive Banach space is $\mathcal{F}_\sigma$. In particular, the point spectrum of a closed densely defined operator in a separable Hilbert space is $\mathcal{F}_\sigma$.*

The second statement of the above result answers in the affirmative the question posed by Jung and Stochel in [7].

3.8. *Remark.* Corollary 3.7 for bounded operators may also be proved in a different way. Namely, if T is a bounded operator on a separable reflexive Banach space X (respectively a closed densely defined operator in a separable Hilbert space $\mathcal{H}$), then, by the reflexivity of X, $\mathcal{N}(T) = \{0\}$ iff the range of the dual operator $T'\colon X' \to X'$ (respectively iff the range of the adjoint operator T^*) is dense in X' (in $\mathcal{H}$). So, $\sigma_p(T)$ consists of precisely those $\lambda \in \mathbb{K}$ for which $\mathcal{R}(S_\lambda)$ is **not** dense in X' (in $\mathcal{H}$) where $S_\lambda = T' - \lambda$ ($S_\lambda = T^* - \bar{\lambda}$). Consequently, if

$\{y_n\}_{n=1}^{\infty}$ is a dense sequence in X' (in $\mathcal{H}$), then

$$\sigma_p(T) = \bigcup_{n,m\geqslant 1} \bigcap_{x\in\mathcal{D}(S_0)} F(n,m,x)$$

with $F(n,m,x) := \{\lambda \in \mathbb{K}\colon\ \|S_\lambda x - y_n\| \geqslant \frac{1}{m}\}$. We infer from the closedness of $F(n,m,x)$'s that indeed $\sigma_p(T)$ is $\mathcal{F}_\sigma$. This elementary and simpler proof does not work in case when T is only closable. Moreover, this approach hides the crucial property of reflexivity for this issue revealed by Lemma 3.1, namely the weak σ-compactness of the domain of an operator.

3.9. **Example.** Let $\mathcal{H}$ be a complex separable infinite-dimensional Hilbert space and let $S\colon \mathcal{H} \to \mathcal{H}$ be the backward shift with respect to the orthonormal basis $\mathcal{E} = (e_n)_{n=0}^{\infty}$ of $\mathcal{H}$; that is, $Se_0 = 0$ and $Se_n = e_{n-1}$ for $n \geqslant 1$. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\psi\colon \mathbb{D} \ni z \mapsto \sum_{n=0}^{\infty} z^n e_n \in \mathcal{H}$. Recall that $\sigma_p(S) = \mathbb{D}$, $S\psi(z) = z\psi(z)$ for each $z \in \mathbb{D}$ and the image of ψ is linearly independent. What is more, the linear span $\mathcal{D}_F$ of $\psi(F)$ is dense in $\mathcal{H}$ for every uncountable subset F of $\mathbb{D}$. (To see the latter, note that if x is orthogonal to $\psi(F)$, then $\sum_{n=0}^{\infty} x_n z^n = 0$ for each $z \in F$ where x_n is the n-th Fourier coefficient of x in the basis $\mathcal{E}$. Thus $x_n = 0$ for any n and $x = 0$ as well.) So, the restriction S_F of S to $\mathcal{D}_F$ is a bounded densely defined operator in $\mathcal{H}$ such that $\sigma_p(S_F) = F$. The example shows that in Theorem 3.5 the domain of a closable operator has to have additional properties beside the separability.

3.10. **Example.** The inclusion map of $L^\infty[0,1]$ into $L^2[0,1]$, as the dual operator of the inclusion map of $L^2[0,1]$ into $L^2[0,1]$, is continuous with respect to the weak-* topology of the domain and the weak one in $L^2[0,1]$. This implies that $L^\infty[0,1]$, considered as a subspace of $L^2[0,1]$, is weakly σ-compact. However, there is no closed operator in $L^2[0,1]$ whose domain is $L^\infty[0,1]$ (compare with the remark on page 257 in [5]). The example shows that Theorem 3.5 can have quite natural applications also for nonclosed operators and that it is more general than Corollary 3.6 which does not apply here, since there is no bounded operator on a reflexive Banach space into $L^2[0,1]$ whose image is $L^\infty[0,1]$ (again, by the remark on page 257 in [5]).

4. Decomposition of the point spectrum

Let T be an operator in X. For $n \geqslant 1$ let $\sigma_{p[n]}(T)$ be the set of all $\lambda \in \mathbb{K}$ such that $\dim \mathcal{N}(T-\lambda) = n$ and let $\sigma_{p[\infty]}(T) = \sigma_p(T) \setminus \bigcup_{n=1}^{\infty} \sigma_{p[n]}(T)$. Our aim is to show that all just defined sets are Borel provided T is closable and has separable and weakly σ-compact domain. To do this, we need

4.1. **Lemma.** *If V is a T_2 topological vector space, then for each $n \geqslant 2$ the set $F[n]$ of all $(x_1,\dots,x_n) \in V^n$ such that $x_1,\dots,x_n$ are linearly dependent is closed in the product topology of V^n.*

Proof. Let $\Delta = \{\lambda \in \mathbb{K}\colon\ |\lambda| \leqslant 1\}$. For a permutation τ of $\{1,\dots,n\}$ let F_τ be the set of all n-tuples $(x_1,\dots,x_n) \in V^n$ such that $x_{\tau(1)} = \sum_{j=2}^n \lambda_j x_{\tau(j)}$ for some $\lambda_2,\dots,\lambda_n \in \Delta$. Notice that $F[n] = \bigcup_{j=1}^n F_{\tau_j}$ where $\tau_j(1) = j$, $\tau_j(j) = 1$ and $\tau_j(k) = k$ for $k \neq 1, j$. So, it suffices to show that F_τ is closed. But $F_\tau = p(\Phi^{-1}(\{0\})$ where $p\colon \Delta^{n-1} \times V^n \to V^n$ is the projection onto the second factor and

$$\Phi\colon \Delta^{n-1} \times V^n \ni (\lambda_2,\dots,\lambda_n; x_1,\dots,x_n) \mapsto x_1 - \sum_{j=2}^n \lambda_j x_j \in V.$$

Since Δ^{n-1} is compact, p is a closed map and we are done. $\square$

The main result of the section is the following

4.2. **Theorem.** *If T is a closable operator in X whose domain is separable and weakly σ-compact, then $\sigma_{p[n]}(T)$ is $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$ for finite n and $\sigma_{p[\infty]}(T)$ is $\mathcal{F}_{\sigma\delta}$.*

Proof. First fix finite $N > 0$. Let $F[1] = \{0\} \subset X$ and $F[N]$ be as in the statement of Lemma 4.1 for $N > 1$. By Lemma 4.1, $F[N]$ is weakly closed in X^N. Write $\mathcal{D}(T) = \bigcup_{n=1}^\infty K_n$ with each K_n weakly compact. By Lemma 3.3, all K_n's are weakly metrizable and hence $(\prod_{j=1}^N K_{n_j}) \setminus F[N]$ is $\mathcal{F}_\sigma$ in $\prod_{j=1}^N K_{n_j}$ for any $n_1,\dots,n_N$ when each K_n is equipped with the weak topology. So, $\mathcal{D}(T)^N \setminus F[N]$ may be written in the form

$$\mathcal{D}(T)^N \setminus F[N] = \bigcup_{n=1}^\infty L_n \tag{4-1}$$

where each L_n is a weakly compact subset of X^N. Put $S\colon \mathcal{D}(T)^N \ni (x_1,\dots,x_N) \mapsto (Tx_1,\dots,Tx_N) \in X^N$ and observe that S is a closable operator in X^N. Thanks to Lemma 3.1, the set $\Lambda_S(L_n, m)$ is compact for all natural n and m. So, $G_N := \bigcup_{n,m} \Lambda_S(L_n, m)$ is $\mathcal{F}_\sigma$. But G_N is the set of all $\lambda \in \sigma_p(T)$ such that $\mathcal{N}(T-\lambda)$ is at least$N$-dimensional, by (4-1).

Now observe that for finite n, $\sigma_{p[n]}(T) = G_n \setminus G_{n+1}$ and thus $\sigma_{p[n]}(T)$ is $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$. Finally, since $\sigma_{p[\infty]}(T) = \bigcap_{n=1}^\infty G_n$, $\sigma_{p[\infty]}(T)$ is $\mathcal{F}_{\sigma\delta}$. $\square$

5. Decomposition of the spectrum: Hilbert space

From now on, $\mathcal{H}$ is a complex separable infinite-dimensional Hilbert space and $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$. Every subset of $\mathcal{B}(\mathcal{H})$ which is Borel with respect to, respectively, the weak operator, the strong operator or the norm topology is called by us, respectively, *WOT-Borel*, *SOT-Borel* and shortly *Borel*. Similarly, if $f\colon S \to \mathcal{B}(\mathcal{H})$ with $S \subset \mathcal{B}(\mathcal{H})$

is such that S and the inverse image of any WOT-Borel (respectively SOT-Borel; Borel) set is WOT-Borel (SOT-Borel; Borel) is said to be *WOT-Borel* (*SOT-Borel*; *Borel*). A fundamental result in this topic says that every WOT-Borel set is SOT-Borel and conversely (see e.g. [4]). Thus the same property holds true for WOT-Borel and SOT-Borel functions. Therefore we shall only speak on WOT-Borel and Borel sets and functions. Whenever in the sequel the classes $\mathcal{F}_\sigma$, $\mathcal{G}_\delta$, $\mathcal{F} \cap \mathcal{G}$, etc., appear, they are understood with respect to the norm topology.

We are interested in the decomposition of $\mathcal{B}(\mathcal{H})$ into Borel sets each of which collects the operators of a similar type. For $n = 0, 1, 2, \ldots$ let $\Sigma_n(\mathcal{H})$ be the set of all operators $A \in \mathcal{B}(\mathcal{H})$ such that $\dim \mathcal{R}(A) = n$. Further, for $n, m = 0, 1, 2, \ldots, \infty$ let, respectively, $\Sigma^1_{n,m}(\mathcal{H})$ and $\Sigma^0_{n,m}(\mathcal{H})$ consist of all $A \in \mathcal{B}(\mathcal{H})$ such that $\dim \mathcal{N}(A) = n$, $\dim \mathcal{R}(A)^\perp = m$, $\dim \overline{\mathcal{R}}(A) = \infty$ and, respectively, $\mathcal{R}(A)$ is closed or not. Notice that all just defined sets of operators form a countable decomposition of $\mathcal{B}(\mathcal{H})$, denoted by $\Sigma(\mathcal{H})$, into nonempty and pairwise disjoint sets. We want to show that each of them is Borel in $\mathcal{B}(\mathcal{H})$. To do this, we need

5.1. **Lemma.** *For each $n = 0, 1, 2, \ldots, \infty$, let $\Sigma_{n,*}(\mathcal{H})$ be the set of all operators $A \in \mathcal{B}(\mathcal{H})$ such that* $\dim \mathcal{N}(A) = n$. *Then $\Sigma_{0,*}(\mathcal{H})$ is $\mathcal{G}_\delta$, $\Sigma_{n,*}(\mathcal{H})$ is $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$ for finite $n > 0$ and $\Sigma_{\infty,*}(\mathcal{H})$ is $\mathcal{F}_{\sigma\delta}$ in $\mathcal{B}(\mathcal{H})$.*

Proof. We mimic the proof of Theorem 4.2. For fixed finite $N > 0$ let $F[N]$ be defined as there. Write $\mathcal{H}^N \setminus F[N] = \bigcup_{n=1}^\infty L_n$ with each L_n weakly compact in $\mathcal{H}^N$. For $T \in \mathcal{B}(\mathcal{H})$ let $T^{\times N} \in \mathcal{B}(\mathcal{H}^N)$ denote the operator $\mathcal{H}^N \ni (x_1, \ldots, x_N) \mapsto (Tx_1, \ldots, Tx_N) \in \mathcal{H}^N$. Since L_n is bounded, the function $\psi\colon \mathcal{B}(\mathcal{H}) \times L_n \ni (T, x) \mapsto T^{\times N} x \in \mathcal{H}^N$ is continuous when $\mathcal{B}(\mathcal{H})$ is considered with the norm topology and L_n and $\mathcal{H}^N$ with the weak one. Hence $\psi^{-1}(\{0\})$ is closed in $\mathcal{B}(\mathcal{H}) \times L_n$ in the product topology of these topologies. Finally, since L_n is weakly compact, the set $F_n := p(\psi^{-1}(\{0\}))$ is closed in the norm topology of $\mathcal{B}(\mathcal{H})$ where $p\colon \mathcal{B}(\mathcal{H}) \times L_n \to \mathcal{B}(\mathcal{H})$ is the projection onto the first factor. Thus $G_N := \bigcup_{n=1}^\infty F_n$ is $\mathcal{F}_\sigma$. Note that G_N coincides with the set of all $A \in \mathcal{B}(\mathcal{H})$ for which $\dim \mathcal{N}(A) \geqslant N$.

Now we have $\Sigma_{n,*}(\mathcal{H}) = G_n \setminus G_{n+1}$ for finite $n > 0$, $\Sigma_{\infty,*}(\mathcal{H}) = \bigcap_{n=1}^\infty G_n$ and $\Sigma_{0,*}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \setminus G_1$ which clearly finishes the proof. □

Now let $\mathcal{CR}(\mathcal{H})$ be the set of all closed range operators of $\mathcal{B}(\mathcal{H})$. Our next purpose is to show that $\mathcal{CR}(\mathcal{H})$ is WOT-Borel (and hence Borel as well). This is however not so simple. Firstly, the maps $\mathcal{B}(\mathcal{H}) \ni A \mapsto A^* \in \mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \ni (A, B) \mapsto AB \in \mathcal{B}(\mathcal{H})$ are WOT-Borel (cf. [4]), since the first of them is WOT-continuous and the second is SOT-continuous on bounded sets. Secondly, closed range operators may be characterized in the space $\mathcal{B}(\mathcal{H})$ by means of the so-called *Moore-Penrose inverse* in the following way:

5.2. **Proposition.** *An operator $A \in \mathcal{B}(\mathcal{H})$ has closed range iff there is $B \in \mathcal{B}(\mathcal{H})$ such that $ABA = A$, $BAB = B$ and AB and BA are selfadjoint. What is more, the operator B is uniquely determined by these properties and if A has closed range, $B = A^\dagger := (A\big|_{\mathcal{R}(A^*)})^{-1}P$ where P is the orthogonal projection onto the range of A.*

The operator $A^\dagger$ appearing in the statement of Proposition 5.2 is the above mentioned Moore-Penrose inverse of an operator $A \in \mathcal{CR}(\mathcal{H})$. For the proof of Proposition 5.2, see e.g. [12].

In the proof of the next result we use the theorem on Souslin sets which states that if Y and Z are separable complete metric spaces, B is a Borel subset of Y and $f\colon B \to Z$ is a continuous one-to-one function, then $f(B)$ is a Borel subset of Z (see [8, Theorem XIII.1.9] or [15, Corollary A.7], or [16, Theorem A.25] for more general result).

5.3. **Theorem.** *The set $\mathcal{CR}(\mathcal{H})$ is WOT-Borel.*

Proof. For $r > 0$ let B_r be the closed ball in $\mathcal{B}(\mathcal{H})$ with center at 0 and of radius r equipped with the weak operator topology. Let $\psi_r\colon B_r^2 \ni (T,S) \mapsto (TST - T, STS - S, S^*T^* - TS, T^*S^* - ST) \in B_s^4$ where $s := 2(r+1)^3$. By the note preceding the statement of the theorem, ψ_r is WOT-Borel. So, the set $C_r := \psi_r^{-1}(\{(0,0,0,0)\})$ is WOT-Borel in B_r as well. By Proposition 5.2, $C_r = \{(A, A^\dagger)\colon\ A \in \mathcal{CR}(\mathcal{H}),\ A, A^\dagger \in B_r\}$. So, the projection p_r of C_r onto the first factor is one-to-one. But p_r is WOT-continuous and B_r and B_s are compact metrizable spaces. We infer from this that $E_r := p_r(C_r)$ is WOT-Borel in B_s. Finally, the observation that B_s is WOT-Borel in $\mathcal{B}(\mathcal{H})$ and $\mathcal{CR}(\mathcal{H}) = \bigcup_{n=1}^\infty E_n$ finishes the proof. $\square$

Question 2. Which additive or multiplicative class, in the hierarchy of WOT-Borel sets in $\mathcal{B}(\mathcal{H})$, the set $\mathcal{CR}(\mathcal{H})$ is of?

Now we are ready to prove the following

5.4. **Theorem.** *For each $k, n, m = 0, 1, 2, \ldots, \infty$ with k finite:*

(a) *$\Sigma_k(\mathcal{H})$ is $\mathcal{F} \cap \mathcal{G}$,*
(b) *$\Sigma^1_{n,m}(\mathcal{H})$ is $\mathcal{F} \cap \mathcal{G}$ (respectively open) provided m or n is finite (respectively $m = 0$ or $n = 0$),*
(c) *$\Sigma^0_{0,0}(\mathcal{H})$ is $\mathcal{G}_\delta$; $\Sigma^0_{n,m}(\mathcal{H})$ is $\mathcal{F}_\sigma \cap \mathcal{G}_\delta$ for finite n and m; $\Sigma^0_{n,m}(\mathcal{H})$ is $\mathcal{F}_{\sigma\delta}$ if either n or m is infinite (and the other is finite),*
(d) *$\Sigma^1_{\infty,\infty}(\mathcal{H})$ and $\Sigma^0_{\infty,\infty}(\mathcal{H})$ are Borel.*

Proof. Clearly, for each finite k the set of all finite rank operators $A \in \mathcal{B}(\mathcal{H})$ such that $\dim \mathcal{R}(A) \leqslant k$ is closed and therefore $\Sigma_k(\mathcal{H})$ is $\mathcal{F} \cap \mathcal{G}$. Further, for each $n = 0, 1, 2 \ldots, \infty$ let $\Sigma_{n,*}(\mathcal{H})$ be as in Lemma 5.1 and let $\Sigma_{*,n}(\mathcal{H}) = \{A^*\colon\ A \in \Sigma_{n,*}(\mathcal{H})\}$. Observe that $\Sigma_{*,n}(\mathcal{H})$ is of the same Borel class as $\Sigma_{n,*}(\mathcal{H})$.

Suppose that n or m is finite. With no loss of generality, we may assume that $n \leqslant m$. Put $k = n - m \in \mathbb{Z} \cup \{-\infty\}$. The set $F(k)$ of all semi-Fredholm operators of index k is open in $\mathcal{B}(\mathcal{H})$ (see e.g. Proposition XI.2.4 and Theorem XI.3.2 in [2]). It may also be shown that for each integer $l \geqslant 0$ the set $F_l(k)$ of all $A \in F(k)$ such that $\dim \mathcal{N}(A) \leqslant l$ is open in $\mathcal{B}(\mathcal{H})$ as well (see e.g. [11, Proposition 5.3]). Now the relation $\Sigma^1_{n,m}(\mathcal{H}) = F_n(k) \setminus F_{n-1}(k)$ (with $F_{-1}(k) = \varnothing$) shows (b). Further, since $\Sigma^0_{n,m}(\mathcal{H}) = \Sigma_{n,*}(\mathcal{H}) \cap \Sigma_{*,m}(\mathcal{H}) \setminus F(k)$, we infer from Lemma 5.1 the assertion of (c).

Finally, $\Sigma^1_{\infty,\infty}(\mathcal{H}) = \Sigma_{\infty,*}(\mathcal{H}) \cap \Sigma_{*,\infty}(\mathcal{H}) \cap \mathcal{CR}(\mathcal{H}) \setminus \bigcup_{n=0}^{\infty} \Sigma_n(\mathcal{H})$. So, this set is Borel by Theorem 5.3. Since $\Sigma^0_{\infty,\infty}(\mathcal{H})$ is the complement in $\mathcal{B}(\mathcal{H})$ of the union of all other members of $\Sigma(\mathcal{H})$, it is Borel as well. $\square$

Question 3. Which additive or multiplicative class, in the hierarchy of Borel sets in $\mathcal{B}(\mathcal{H})$, the sets $\Sigma^1_{\infty,\infty}(\mathcal{H})$ and $\Sigma^0_{\infty,\infty}(\mathcal{H})$ are of?

Now let T be a closed densely defined operator in $\mathcal{H}$. By $\sigma(T)$ we denote the spectrum of T. That is, a complex number λ does **not** belong to $\sigma(T)$ iff $\mathcal{N}(T-\lambda) = \{0\}$, $\mathcal{R}(T-\lambda) = \mathcal{H}$ and $(T-\lambda)^{-1}$ is bounded (the latter condition may be omitted by the Closed Graph Theorem). We decompose the complex plane into parts corresponding to the members of $\Sigma(\mathcal{H})$:

- $\sigma^f(T)$ is the set of all $z \in \mathbb{C}$ such that $\mathcal{R}(T-z)$ is finite-dimensional,
- for $n, m = 0, 1, 2, \ldots, \infty$ let $\sigma^1_{n,m}(T)$ and $\sigma^0_{n,m}(T)$ be the sets consisting of all $z \in \mathbb{C}$ for which $\dim \mathcal{N}(T-z) = n$, $\dim \mathcal{R}(T-z)^{\perp} = m$, $\dim \overline{\mathcal{R}}(T-z) = \infty$ and, respectively, $\mathcal{R}(T-z)$ is closed or not.

Notice that $\mathbb{C} \setminus \sigma(T) = \sigma^1_{0,0}(T)$ is the resolvent set of T and that the above defined sets are pairwise disjoint and cover the complex plane. The collection of all of them is denoted by $\Sigma(T)$. We say that the sets $\sigma^1_{n,m}(T)$ and $\sigma^0_{n,m}(T)$ correspond to, respectively, $\Sigma^1_{n,m}(\mathcal{H})$ and $\Sigma^0_{n,m}(\mathcal{H})$. What we want is to prove

5.5. **Proposition.** *For every closed densely defined operator T in $\mathcal{H}$, $\Sigma(T)$ consists of Borel subsets of $\mathbb{C}$. What is more,* $\operatorname{card} \sigma^f(T) \leqslant 1$ *and each member of $\Sigma(T)$ different from $\sigma^f(T)$ is of the same Borel class as the corresponding member of $\Sigma(\mathcal{H})$.*

Proof. First observe that if $z \in \sigma^f(T)$, then $\mathcal{D}(T) = \mathcal{H}$ and T is bounded (since $\mathcal{N}(T-z)$ is closed). We conclude from this that indeed $\operatorname{card} \sigma^f(T) \leqslant 1$. Further, since T is closed, there is an operator $A \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = \mathcal{D}(T)$ (e.g. $A := (|T|+1)^{-1}$; see also [5, Theorem 1.1]). Put $C = TA \in \mathcal{B}(\mathcal{H})$. Notice that for each $z \in \mathbb{C}$, $\mathcal{R}(C - zA) = \mathcal{R}(T-z)$ and $\mathcal{N}(C - zA) = A^{-1}(\mathcal{N}(T-z))$, and thus $\dim \mathcal{N}(T-z) = \dim \mathcal{N}(C - zA)$. It is infered from this that

$z \in \sigma^j_{n,m}(T)$ iff $C - zA \in \Sigma^j_{n,m}(\mathcal{H})$. So, the continuity of the function $\mathbb{C} \ni z \mapsto C - zA \in \mathcal{B}(\mathcal{H})$ finishes the proof. □

We end the paper with notes concerning the so-called *residual* and *continuous* spectra of an operator. In the literature there are at least two different, nonequivalent definitions of them. For some mathematicians the residual spectrum $\sigma_r(T)$ of a closed densely defined operator T in a separable Hilbert space $\mathcal{H}$ consists of all $\lambda \in \sigma(T) \setminus \sigma_p(T)$ such that $\mathcal{R}(T - \lambda)$ is closed (e.g. [1]), for others it is the set of all $\lambda \in \sigma(T) \setminus \sigma_p(T)$ such that $\overline{\mathcal{R}}(T - \lambda) \neq \mathcal{H}$ (e.g. [10]). Similarly, the continuous spectrum $\sigma_c(T)$ of T for some people means the set of all $\lambda \in \mathbb{K}$ such that $\mathcal{R}(T - \lambda)$ is nonclosed (e.g. [1]), for others it coincides with $\sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T))$ (e.g. [10]). Somehow or other — whatever definitions we choose among the above, both the residual and the continuous spectra are Borel and are the unions of subfamilies of $\Sigma(T)$.

References

[1] M.S. Birman and M.Z. Solomjak, *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, D. Reidel Publishing Co., Dordrecht, 1987.

[2] J.B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, 1985.

[3] M.J. Cowen and R.G. Douglas, *Complex geometry and operator theory*, Acta Math. **141** (1978), 187–261.

[4] E.G. Effros, *Global structure in von Neumann algebras*, Trans. Amer. Math. Soc. **121** (1966), 434–454.

[5] P.A. Fillmore and J.P. Williams, *On operator ranges*, Adv. in Math. **7** (1971), 254–281.

[6] C. Jiang, *Similarity classification of Cowen-Douglas operators*, Canad. J. Math. **56** (2004), 742–775.

[7] I.B. Jung and J. Stochel, *Subnormal operators whose adjoints have rich point spectrum*, J. Funct. Anal. **255** (2008), 1797–1816.

[8] K. Kuratowski and A. Mostowski, *Set Theory with an Introduction to Descriptive Set Theory*, PWN – Polish Scientific Publishers, Warszawa, 1976.

[9] J.E. McCarthy, *Boundary values and Cowen-Douglas curvature*, J. Funct. Anal. **137** (1996), 1–18.

[10] W. Mlak, *Hilbert Spaces and Operator Theory*, PWN — Polish Scientific Publishers and Kluwer Academic Publishers, Warszawa-Dordrecht, 1991.

[11] P. Niemiec, *Norm closures of orbits of bounded operators*, to appear.

[12] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Phil. Soc. **51** (1955), 406–413.

[13] J. Stochel and F.H. Szafraniec, *On normal extensions of unbounded operators. III. Spectral properties*, Publ. Res. Inst. Math. Sci. **25** (1989), 105–139.

[14] A.H. Stone, *Absolute $\mathcal{F}_\sigma$-spaces*, Proc. Amer. Math. Soc. **13** (1962), 495–499.

[15] M. Takesaki, *Theory of Operator Algebras I* (Encyclopaedia of Mathematical Sciences, Volume 124), Springer-Verlag, Berlin-Heidelberg-New York, 2002.

[16] M. Takesaki, *Theory of Operator Algebras III* (Encyclopaedia of Mathematical Sciences, Volume 127), Springer-Verlag, Berlin-Heidelberg-New York, 2003.

[17] K. Zhu, *Operators in Cowen-Douglas classes*, Illinois J. Math. **44** (2000), 767–783.

Piotr Niemiec, Jagiellonian University, Institute of Mathematics, ul. Łojasiewicza 6, 30-348 Kraków, Poland

E-mail address: piotr.niemiec@uj.edu.pl